A LIMIT THEOREM FOR RANDOM COVERINGS OF A CIRCLE

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ABSTRACT

Let $N_{\alpha,m}$ equal the number of randomly placed arcs of length α $(0 < \alpha < 1)$ required to cover a circle C of unit circumference m times. We prove that $\lim_{\alpha\to 0} P(N_{\alpha,m} \leqq (1/\alpha)(\log(1/\alpha) + m\log\log(1/\alpha) + x) = \exp((-1/(m-1)!)\exp(-x))$. Using this result for m = 1, we obtain another derivation of Steutel's result $E(N_{\alpha,1}) = (1/\alpha)(\log(1/\alpha) + \log\log(1/\alpha) + \gamma + o(1))$ as $\alpha \to 0$, γ denoting Euler's constant.

1.

Let C be a circle of unit circumference. Suppose that arcs of given length α $(0 < \alpha < 1)$ are thrown independently and uniformly on C. The distribution function of the number N_α of these randomly placed arcs needed to cover the circle C has been calculated by Stevens [9] who has shown that

$$P(N_\alpha \leqq n) = \sum_{0 \leqq k \leqq 1/\alpha} (-1)^k \binom{n}{k} (1 - k\alpha)^{n-1} \tag{1.1}$$

for any positive integer n.

Using (1.1), one may readily compute the expectation $E(N_\alpha)$ as

$$E(N_\alpha) = 1 - \sum_{1 \leqq k \leqq 1/\alpha} (-1)^k \frac{(1 - k\alpha)^{k-1}}{(k\alpha)^{k+1}} \tag{1.2}$$

(a derivation of (1.2) is given in [5]).

Unfortunately, neither (1.1) nor (1.2) is very illuminating, since the summands undergo violent oscillations; therefore, it becomes of interest to study the asymptotic behavior of $P(N_\alpha \leqq n)$, $E(N_\alpha)$ as $\alpha \to 0$. Using (1.2), Flatto and Konheim [5] have shown that

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$$E(N_\alpha) \sim \frac{1}{\alpha}\log\frac{1}{\alpha}. \tag{1.3}$$

This result has subsequently been improved by Steutel [8] who, using Laplace transform methods, obtained

$$E(N_\alpha) = \frac{1}{\alpha}\left(\log\frac{1}{\alpha} + \log\log\frac{1}{\alpha} + \gamma + o(1)\right) \quad \text{as } \alpha \to 0 \tag{1.4}$$

where γ is Euler's constant.

The problem of describing the asymptotic behavior of $P(N_\alpha \leqq n)$ has recently been studied by Shepp [7]. He proved that the random variable $\alpha(N_\alpha - (1/\alpha)(\log(1/\alpha) + \log\log(1/\alpha))$ has a proper limiting distribution as $\alpha \to 0$, and he obtained estimates for the tails of this distribution.† The problem of obtaining the exact expression for this limiting distribution is left open in Shepp's paper. We shall obtain this limiting distribution in the present paper. In fact, we derive the following more general result in Section 2.

THEOREM 2.6. *Let $N_{\alpha,m}$ equal the number of randomly placed arcs of length α required to cover the circle C m times. Then*

$$\lim_{\alpha\to 0} P\left(N_{\alpha,m} \leqq \frac{1}{\alpha}\left(\log\frac{1}{\alpha} + m\log\log\frac{1}{\alpha} + x\right)\right) = e^{-1/(m-1)!e^{-x}} .$$

The limiting distribution of Theorem 2.6 is one of the extreme value distributions obtained by Gnedenko in his theory of the limiting distribution of the maximum term in a sequence of identically distributed independent random variables [6]. We give a heuristic derivation of Theorem 2.6 which brings out clearly the connection between the latter and the Gnedenko Theory. It suffices to consider the case $\alpha = 1/n$, n a positive integer $\to \infty$, as the general result is easily deduced from this particular case. We divide the circle C into n equal arcs which we label $C_1, \dots, C_n$. Let N_i = number of throws necessary to cover C_i m times. Since C is covered m times iff each C_i is covered m times, we have $N_{1/n,m} = \max(N_1, \dots, N_n)$. Let $N = N(n)$ be a positive integer so that $\lim_{n\to\infty}(N/n) = \infty$. A somewhat lengthy calculation, which we omit, shows that

† The proof in [7] is erroneous. Shepp obtained upper and lower estimates for $\overline{\lim}$, $\underline{\lim}$ of $P(N_\alpha - (1/\alpha)(\log(1/\alpha) + \log\log(1/\alpha)) > x)$ as $\alpha \to 0$ (formula (94) of [7]). From these alone, we cannot conclude the existence of the limiting distribution. The estimates of course provide bounds for the tails of the limiting distribution, provided that we know independently that the latter exists.

(1.5) $$P(N_i > N) \sim \frac{1}{(m-1)!}\left(\frac{N}{n}\right)^m\left(1-\frac{1}{n}\right)^N \text{ as } n \to \infty \text{ for } 1 \leqq i \leqq n.$$

We now choose $N = [N']$ where $N' = n(\log n + m\log\log n + x)$, $[N']$ denoting, as usual, the greatest integer $\leqq N'$. It is readily checked that $N/n \sim \log n$ $(1-1/n)^N \sim e^{-x}/n(\log n)^m$. Thus (1.5) becomes

(1.6) $$P(N_i > N') = P(N_i \geqq N) \sim \frac{e^{-x}}{(m-1)!\,n}.$$

Now $P(N_{1/n,m} \leqq N') = P(N_1 \leqq N', \cdots, N_n \leqq N')$. Treating the N_i's as "independent random variables", we obtain

(1.7)

$$P(N_{1/n,m} \leqq N') = \prod_{i=1}^{n} P(N_i \leqq N') = \prod_{i=1}^{n} (1 - P(N_i > N')) \sim \left(1 - \frac{e^{-x}}{(m-1)!n}\right)^n$$

$$\sim e^{\frac{-e^{-x}}{(m-1)!}}.$$

The above argument is only heuristic because the N_i's are not independent random variables. Thus the Gnedenko Theory, which deals only with independent random variables, does not apply as it stands to our problem. We provide another heuristic explanation of Theorem 2.6 in Section 2.

Folllowing a suggestion of Shepp, we show in Section 3 how Steutel's asymptotic formula (1.4) for $E(N_\alpha)$ follows from Theorem 2.6. To do this, we shall require estimates on the tails of the distribution of $\alpha\ (N_\alpha - (1/\alpha)(\log 1/\alpha + \log\log(1/\alpha)))$ which is uniform in α. Finally, we point out in Section 3 the analogy between Theorem 2.6 and a result of Erdös and Renyi [2], who have studied a discrete analog of our problem.

2.

We proceed to prove Theorem 2.6 which we stated in Section 1. First we obtain an interpretation of the quantity $P(N_{\alpha,m} \leqq n)$, which proves to be useful.

THEOREM 2.1. *Let $n-1$ $(n > m)$ points be chosen independently and uniformly in the interval $[0,1]$. These $n-1$ points partition $[0,1]$ into n intervals. Let $L_0, L_1, \cdots, L_{n-1}$ denote the lengths of the successive intervals. Define L_i for all i by the requirement $L_{i+n} = L_i$ and set $S_i = L_i + \cdots + L_{i+m-1}$. Then $P(N_{\alpha,m} \leqq n) = P(S_i \leqq \alpha,\ 0 \leqq i \leqq n-1)$.*

REMARKS

1) Strictly speaking, we should write for fixed m, $L_i^{(n)}, S_i^{(n)}$ instead of L_i, S_i since these random variables depend both on i and n. We shall employ

the superscript n in some of the ensuing theorems where it becomes important to distinguish different values of n.

2) The event $[N_{\alpha,m} \leqq n]$ means that the first n randomly placed arcs cover C at least m times. The probability of this event is the same regardless of whether the arcs are open or closed. This follows easily from the fact that the probability that two of the arcs are contiguous is 0. For definiteness, we assume in the ensuing proof that the arcs are closed.

PROOF OF THEOREM 2.1. We give C a fixed orientation so that it becomes meaningful to speak of a left and right end-point of an arc placed on C. We denote the closed arc with left and right end-points α, β as $[\alpha,\beta]$; similarly, an open arc with end-points α,β is denoted as (α,β). We use $L(\alpha,\beta)$ to denote the length of either arc. The positions of the first n randomly placed arcs are determined by their left end-points. We label the arcs as $[p_0,q_0],\cdots,[p_{n-1},q_{n-1}]$ where $p_0,p_1,\cdots,p_{n-1}$ are consecutive points on C consistent with its orientation and $[p_0,q_0]$ is the firs arc placed on C. Define p_i,q_i for all i by the requirement $p_i = p_{i+n}, q_i = q_{i+n}$. The points $p_0,p_1,\cdots,p_{n-1}$ partition C into n arcs $[p_i,p_{i+1}]$, $0 \leqq i \leqq n-1$.

The requirement that C be covered at least m times by $[p_i,q_i]$, $0 \leqq i \leqq n-1$, is equivalent to demanding that $L(p_i,p_{i+m}) \leqq \alpha$ for all i. For suppose that the latter holds; we then have $L(p_j,p_{i+m}) \leqq \alpha$ for $i \leqq j \leqq i+m-1$. Hence $[p_{i+m-1},p_{i+m}] \subseteq [p_j,p_{i+m}] \subseteq [p_j,q_j]$ for $i \leqq j \leqq i+m-1$. Since each $p \in C$ belongs to some $[p_{i+m-1},p_{i+m}]$, we conclude that each $p \in C$ is covered by at least m of the n arcs $[p_j,q_j]$, $0 \leqq j \leqq n-1$. Conversely, suppose all points of C are covered by at least m of the arcs $[p_j,q_j]$, $0 \leqq j \leqq n-1$. We conclude that any point $p \in (p_{i+m-1},p_{i+m})$ is covered by $[p_j,q_j]$, $i \leqq j \leqq i+m-1$. In particular $p \in [p_i,q_i]$ so that $L(p_i,p) \leqq \alpha$. Choosing p arbitrarily close to p_{i+m}, we conclude that $L(p_i,p_{i+m}) \leqq \alpha$ for all i.

Choosing p_0 as 0, we open the circle into the interval $[0,1]$. The points $p_1,\dots,p_{n-1}$ become the points $Y_1,\dots,Y_{n-1}$ $(Y_1 \leqq Y_2 \leqq \cdots \leqq Y_{n-1})$ obtained from distributing $n-1$ points randomly in $[0,1]$. The requirement that $L(p_i,p_{i+m}) \leqq \alpha$ becomes equivalent to $S_i \leqq \alpha$, $0 \leqq i \leqq n-1$. Hence

$$P(N_{\alpha,m} \leqq n) = P(S_i \leqq \alpha,\ 0 \leqq i \leqq n-1).$$

REMARKS. In the introductory Section 1, we gave a heuristic derivation of Theorem 2.6. Another heuristic derivation may be based on Theorem 2.1. Let $n = n(\alpha,m,x) = [(1/\alpha)(\log(1/\alpha) + m\log\log(1/\alpha) + x)]$. We conclude from Theorem 2.1 that $P(N_{\alpha,m} \leqq (1/\alpha)(\log(1/\alpha) + m\log\log(1/\alpha) + x)) = P(N_{\alpha,m} \leqq n)$

$= P(S_0^{(n)} \leqq \alpha, \cdots, S_{n-1}^{(n)} \leqq \alpha)$. We prove later on that $P(S_0^{(n)} > \alpha) = \cdots = P(S_{n-1}^{(n)} > \alpha)$ $= \sum_{j=0}^{m-1} \binom{n-1}{j} \alpha^j (1-\alpha)^{n-1-j}$ (formula 2.14), and this sum is readily seen to be

$$\sim \frac{(n\alpha)^{m-1}}{(m-1)!} \sim \frac{e^{-x}}{(m-1)!(1/\alpha)\log(1/\alpha)}.$$

Treating $S_0^{(n)}, \cdots, S_{n-1}^{(n)}$ as "independent random variables", we obtain

$$P\left(N_{\alpha,m} \leqq \frac{1}{\alpha}\left(\log\frac{1}{\alpha} + m\log\log\frac{1}{\alpha} + x\right)\right) = \prod_{i=0}^{n-1} P(S_i^{(n)} \leqq \alpha)$$

$$\sim \left(1 - \frac{e^{-x}}{(m-1)!(1/\alpha)\log(1/\alpha)}\right)^n \sim e^{-e^{-x}/(m-1)!}.$$

We proceed to give a rigorous proof of Theorem 2.6. Writing $P(N_{\alpha,m} \leqq n)$ $= P(S_i \leqq \alpha, 0 \leqq i \leqq n-1) = 1 - P(\bigcup_{i=0}^{n-1}[S_i > \alpha])$ and using the inclusion-exclusion inequalities to estimate $P(\bigcup_{i=0}^{n-1}[S_i > \alpha])$, we obtain for $l \leqq n$

$$(2.1)\qquad \begin{aligned} P(N_{\alpha,m} \leqq n) &\leqq \sum_{k=0}^{l} (-1)^k \sum_{0 \leqq i_1 < \cdots < i_k \leqq n-1} P(S_{i_1}, \cdots, S_{i_k} > \alpha), \ l \text{ even} \\ P(N_{\alpha,m} \leqq n) &\leqq \sum_{k=0}^{l} (-1)^k \sum_{0 \leqq i_1 < \cdots < i_k \leqq n-1} P(S_{i_1}, \cdots, S_{i_k} > \alpha), \ l \text{ odd}. \end{aligned}$$

Let $n = n(\alpha, m, x) = [(1/\alpha)(\log(1/\alpha) + m\log\log(1/\alpha) + x)]$. We have $P(N_{\alpha,m} \leqq (1/\alpha)(\log(1/\alpha) + m\log\log(1/\alpha) + x)) = P(N_{\alpha,m} \leqq n)$ and we use 2.1 to obtain $\lim_{\alpha\to 0} P(N_{\alpha,m} \leqq n)$. We first evaluate for each positive integer k $\lim_{\alpha\to 0} \sum_{0 \leqq i_1 < \cdots < i_k \leqq n-1} P(S_{i_1}, \cdots, S_{i_k} > \alpha)$. We encounter here a basic difference between $m = 1$ and $m > 1$. In the former case $S_i = L_i$ and the random variables $S_0, \cdots, S_{n-1}$ are exchangeable, i.e., all $n!$ permutations $S_{i_1}, \cdots, S_{i_n}$ of $S_0, \cdots, S_{n-1}$ have the same joint distribution function. In particular, all $P(S_{i_1}, \cdots, S_{i_k} > \alpha)$ appearing in the sum $\sum_{0 \leqq i_1 < \cdots < i_k \leqq n-1} P(S_{i_1}, \cdots, S_{i_k} > \alpha)$ are identical. For $m > 1$, this is no longer the case. For instance, a computation shows that for $m = 2$, $P(S_0, S_1 > \alpha) \neq P(S_0, S_2 > \alpha)$. This fact makes the analysis considerably more complicated for $m > 1$. It is instructive first to treat the easy case $m = 1$ and then try to salvage the argument for $m > 1$.

We use the formula

$$(2.2)\qquad P(L_0, \cdots, L_{k-1} > \alpha) = (1 - k\alpha)^{n-1}$$

which is valid whenever $1 \leqq k \leqq n$ and $0 \leqq k\alpha \leqq 1$ [3, p. 42].

Let $n = [(1/\alpha)(\log(1/\alpha) + \log\log(1/\alpha) + x)]$. Since $\log(1-k\alpha) = -k\alpha + O(\alpha^2)$, we have

$$(2.3)\quad (n-1)\log(1-kx) = -k\left(\log\frac{1}{\alpha}+\log\log\frac{1}{\alpha}+x\right)+O\left(\alpha\log\frac{1}{\alpha}\right).$$

Hence

$$(2.4)\quad (1-k\alpha)^{n-1}\sim\left(\frac{1}{\alpha}\log\frac{1}{\alpha}\right)^{-k}e^{-kx}\quad\text{as } \alpha\to 0.$$

The number of k tuples $i_1,\cdots,i_k$ where $0\leqq i_1<\cdots<i_k\leqq n-1$ is

$$\binom{n}{k}\sim\frac{n^k}{k!}\sim\frac{1}{k!}\left(\frac{1}{\alpha}\log\frac{1}{\alpha}\right)^k.$$

It follows that

$$(2.5)\quad \lim_{\alpha\to 0}\sum_{0\leqq i_1<\cdots<i_k\leqq n-1}P(S_{i_1},\cdots,S_{i_k}>\alpha)=\frac{e^{-kx}}{k!}.$$

Using (2.1) and (2.5), we conclude that

$$(2.6)\quad \lim_{\alpha\to 0}P\left(N_{\alpha,1}\leqq\frac{1}{\alpha}\left(\log\frac{1}{\alpha}+\log\log\frac{1}{\alpha}+\frac{x}{\alpha}\right)\right)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}e^{-kx}=e^{-e^{-x}}.$$

We have thus proved Theorem 1.6 for the case $m=1$. We now try to duplicate the above method of proof for $m>1$. We say that the k-tuple $(i_1,\cdots,i_k)$ is m-separated provided $i_{j+1}\geqq i_j+m$ for $1\leqq j\leqq k-1$ and $i_1+n\geqq i_k+m$. The condition of m-separability simply means that the sums $S_{i_1},\cdots,S_{i_k}$ have no common summands. It follows readily from the exchange-ability of $L_0,\cdots,L_{n-1}$ that $P(S_{i_1},\cdots,S_{i_k}>\alpha)$ is identical for all $(i_1,\cdots,i_k)$ which are m-separated. Hence for these k-tuples, $P(S_{i_1},\cdots,S_{i_k}>\alpha)=P(S_0,S_m,\cdots,S_{(k-1)m}<\alpha)$ We evaluate $\lim_{\alpha\to 0}\sum_{0\leqq i_1<\cdots<i_k\leqq n-1}P(S_{i_1},\cdots,S_{i_k}>\alpha)$ in the following manner. We first obtain an asymptotic formula for $P(S_0,S_m,\cdots,S_{(k-1)m}>\alpha)$ as $\alpha\to 0$ (corollary of Theorem 2.2). We then estimate $P(S_{i_1},\cdots,S_{i_k}>\alpha)$ for those k-tuples $(i_1,\cdots,i_k)$ which are not m-separated (Theorem 2.4). It turns out that the contribution to $\sum_{0\leqq i_1<\cdots<i_k\leqq n-1}P(S_{i_1},\cdots,S_{i_k}>\alpha)$ of these k-tuples is negligible as $\alpha\to 0$. The evaluation of $\lim_{\alpha\to 0}\sum_{0\leqq i_1<\cdots<i_k\leqq n-1}P(S_{i_1},\cdots,S_{i_k}>\alpha)$ follows readily from the corollary of Theorem 2.2 and Theorem 2.4 and is carried out in Theorem 2.5.

We first prove a lemma which is required for the proofs of Theorems 2.2 and 2.4. Suppose that $n-1$ $(n>1)$ points $X_1^{(n)},\cdots,X_{n-1}^{(n)}$ are chosen independently and uniformly in $[0,1]$. Let $Y_1^{(n)},\cdots,Y_{n-1}^{(n)}$ be the numbers obtained by rearranging the $X_i^{(n)}$'s in order of increasing magnitude and let $Y_0^{(n)}=0$. Define $Y_i^{(n)}$ for all i by the requirement $Y_{i+n}^{(n)}=Y_i^{(n)}+1$ and let $S_i^{(n)}=Y_{i+m}^{(n)}-Y_i^{(n)}$. Observe that $S_i^{(n)}$ is identical to S_i of Theorem 2.1. We have the following:

LEMMA. *Let* $m \leqq i_1 < \cdots < i_k \leqq n-m$. *Then*

$$(2.7)\quad P(S_0^{(n)}, S_{i_1}^{(n)}, S_{i_2}^{(n)}, \cdots, S_{i_k}^{(n)} > \alpha) = \int_\alpha^1 P\left(S_{i_1-m}^{(n-m)}, S_{i_2-m}^{(n-m)}, \cdots, S_{i_k-m}^{(n-m)} > \frac{\alpha}{1-y}\right) dF(y)$$

where $F(y)$ *is the distribution function of* $S_0^{(n)} = Y_m^{(n)}$.

PROOF. The random variables $Y_1^{(n)}, \cdots, Y_{n-1}^{(n)}$ have the joint probability density function

$$f(y_1, \cdots, y_{n-1}) = \begin{cases} (n-1)! \text{ if } 0 \leqq y_1 \leqq \cdots \leqq y_{n-1} \leqq 1 \text{ [4, p. 386]} \\ 0 \quad \text{otherwise}. \end{cases}$$

Hence for any set of indices $0 \leqq i_0 < i_1 < \cdots < i_k \leqq n-m$

$$(2.8)\quad P(S_{i_0}^{(n)}, S_{i_1}^{(n)}, \cdots, S_{i_k}^{(n)} > \alpha) = (n-1)! \int_R \cdots \int dy_1 \cdots dy_{n-1}$$

where R denotes the region: $0 = y_0 \leqq y_1 \leqq \cdots \leqq y_{n-1}, y_{i_j+m} - y_{i_j} > \alpha$ $(0 \leqq j \leqq k)$. Let $i_0 = 0$. Integrating out the $y_1, \cdots, y_{m-1}$ variables, we may rewrite (2.8) as

$$(2.9)\quad P(S_0^{(n)}, S_{i_1}^{(n)}, S_{i_2}^{(n)}, \cdots, S_{i_k}^{(n)} > \alpha) = \frac{(n-1)!}{(m-1)!} \int_\alpha^1 y_m^{m-1} \left[\int_{R'(y_m)} \cdots \int dy_{m+1} \cdots dy_{n-1}\right] \cdot dy_m$$

where $R'(y_m)$ denotes the region: $y_m \leqq y_{m+1} \leqq \cdots \leqq y_{n-1} \leqq 1$, $y_{i_j+m} - y_{i_j} > \alpha$ $(1 \leqq j \leqq k)$. Let $v_j = (y_{m+j} - y_m)/(1 - y_m)$. Equation (2.9) becomes

$$(2.10)\quad P(S_0^{(n)}, S_{i_1}^{(n)}, S_{i_2}^{(n)}, \cdots, S_{i_k}^{(n)} > \alpha)$$
$$= (n-1)\binom{n-2}{m-1}(n-m-1)! \int_\alpha^1 y_m^{m-1}(1-y_m)^{n-m-1} \cdot \left[\int_{R'} \cdots \int dv_1 \cdots dv_{n-m-1}\right] dy_m$$

where R' denotes the region $0 = v_0 \leqq v_1 \leqq \ldots \leqq v_{n-m-1} \leqq 1$, $v_{i_j} - v_{i_j - m} > \alpha/(1-y_m)$ $(1 \leqq j \leqq k)$.

Using (2.8) with n replaced by $n-m$, we obtain

$$(2.11)\quad P\left(S_{i_1-m}^{(n-m)}, S_{i_2-m}^{(n-m)}, \cdots, S_{i_k-m}^{(n-m)} > \frac{\alpha}{1-y_m}\right) = (n-m-1)! \int_{R'} \cdots \int dv_1 \cdots dv_{n-m-1}.$$

Equations (2.10) and (2.11) yield

$$(2.12)\quad P(S_0^{(n)}, S_{i_1}^{(n)}, S_{i_2}^{(n)}, \cdots, S_{i_k}^{(n)} > \alpha)$$

$$= (n-1)\binom{n-2}{m-1} \int_\alpha^1 P\left(S_{i_1-m}^{(n-m)}, S_{i_2-m}^{(n-m)}, \cdots, S_{i_k-m}^{(n-m)} > \frac{\alpha}{1-y_m}\right) y_m^{m-1}(1-y_m)^{n-m-1} dy_m.$$

Equation (2.12) is identical to (2.7) as $dF(y) = (n-1)\binom{n-2}{m-1}y^{m-1}(1-y)^{n-m-1}dy$ [4, p. 387]. ■

We now prove

THEOREM 2.2. *Let $n-1$ $(n>1)$ points be chosen independently and uniformly in $[0,1]$. Let $km \leqq n$ and $c/n \leqq \alpha \leqq 1$ where c is a positive constant and k,m are positive integers. For fixed k,m,c we have*

$$(2.13)\quad P(S_0^{(n)}, S_m^{(n)}, \cdots, S_{(k-1)m}^{(n)} > \alpha) = \left(\frac{(n\alpha)^{k(m-1)}}{((m-1)!)^k} + O((n\alpha)^{k(m-1)-1})\right) \cdot (1-k\alpha)^{n-1-k(m-1)}.$$

REMARK. For $m=1$, we already stated the exact formula (2.2), $P(L_0^{(n)},\cdots, L_{k-1}^{(n)} > \alpha) = (1-k\alpha)^{n-1}$, which gives more information than (2.13). It is possible to obtain exact formulae for $P(S_0^{(n)}, S_m^{(n)}, \cdots, S_{(k-1)m}^{(n)} > \alpha)$ for all k, m. However, these formulae become progressively more cumbersome with increasing m and formula (2.13) proves to be adequate for our purposes.

PROOF OF THEOREM 2.2. The proof is obtained by an induction on k. For $k=1$, the event $[S_0^{(n)} > \alpha]$ means that at most $(m-1)$ of the randomly chosen points are contained in $[0,\alpha]$. Since the probability of choosing any point in $[0,\alpha]$ is α, we have

$$(2.14)\quad P(S_0^{(n)} > \alpha) = \sum_{j=0}^{m-1} \binom{n-1}{j} \alpha^j (1-\alpha)^{n-1-j}$$
$$= \left(\sum_{j=0}^{m-1} \binom{n-1}{j} \alpha^j (1-\alpha)^{m-1-j}\right)(1-\alpha)^{n-m}.$$

Since $c/n \leqq \alpha \leqq 1$, we have the estimates

$$(2.15)\quad \binom{n-1}{j}\alpha^j(1-\alpha)^{m-1-j} \leqq (n\alpha)^j = O((n\alpha)^{m-2}) \text{ for } 0 \leqq j \leqq m-2$$

$$(2.16)\quad \binom{n-1}{m-1}\alpha^{m-1} = \left(\frac{n^{m-1}}{(m-1)!} + O(n^{m-2})\right)\alpha^{m-1} = \frac{(n\alpha)^{m-1}}{(m-1)!} + O((n\alpha)^{m-2}).$$

Substituting (2.15) and (2.16) into (2.14), we get

$$P(S_0^{(n)} > \alpha) = \left(\frac{(n\alpha)^{m-1}}{(m-1)!} + O((n\alpha)^{m-2})\right) \cdot (1-\alpha)^{n-m},$$

thus proving Theorem 3.2 for $k=1$.

Suppose Theorem 3.1 holds for k. We show that it holds for $k+1$. Assume that $(k+1)\, m \leqq n$, $c/n \leqq \alpha \leqq 1/(k+1)$. Using the above lemma and observing that $P(S_0^{(n-m)}, S_m^{(n-m)}, \cdots, S_{(k-1)m}^{(n-m)} > \alpha/(1-y)) = 0$ for $y > 1 - k\alpha$, we obtain

$$(2.17)\quad P(S_0^{(n)}, S_m^{(n)}, \cdots, S_{km}^{(n)} > \alpha) = (n-1)\binom{n-2}{m-1} \int_\alpha^{1-k\alpha} y^{m-1}(1-y)^{n-m-1} \cdot P\left(S_0^{(n-m)}, \ldots, S_{(k-1)m}^{(n-m)} > \frac{\alpha}{1-y}\right) dy.$$

We have $km \leqq n-m$, $(c/2)/(n-m) \leqq c/n \leqq \alpha/(1-y) \leqq 1/k$. We may therefore apply (2.13) and get

$$(2.18)\quad P\left(S_0^{(n-m)}, \cdots, S_{(k-1)m}^{(n-m)} > \frac{\alpha}{1-y}\right) = \frac{1}{(1-y)^{n-m-1}}\left(\frac{(n\alpha)^{k(m-1)}}{((m-1)!)^k} + O((n\alpha)^{k(m-1)-1})\right) \cdot (1-k\alpha-y)^N \text{ where } N = n-2-(k+1)(m-1).$$

Substituting (2.18) into (2.17), we get

$$(2.19)\quad P(S_0^{(n)}, S_m^{(n)}, \cdots, S_{km}^{(n)} > \alpha) = (n-1)\binom{n-2}{m-1}\left(\frac{(n\alpha)^{k(m-1)}}{((m-1)!)^k} + O((n\alpha)^{k(m-1)-1})\right) \cdot \int_\alpha^{1-k\alpha} y^{m-1}(1-k\alpha-y)^N dy.$$

We evaluate the integral appearing in (2.19). Let $y = x + \alpha$. Then

$$(2.20)\quad \int_\alpha^{1-k\alpha} y^{m-1}(1-k\alpha-y)^N dy = \int_0^{1-(k+1)\alpha} (x+\alpha)^{m-1}(1-(k+1)\alpha-x)^N dx$$
$$= \sum_{j=0}^{m-1}\binom{m-1}{j}\alpha^{m-1-j}\int_0^{1-(k+1)\alpha} x^j(1-(k+1)\alpha-x)^N dx.$$

Let $x = (1-(k+1)\alpha)v$. Then $\int_0^{1-(k+1)\alpha} x^j(1-(k+1)\alpha-x)^N dx = (1-(k+1)\alpha)^{N+j+1}\int_0^1 v^j(1-v)^N dv$. The integral $\int_0^1 v^j(1-v)^N dv$ is recognized to be $B(j+1, N+1) = j!N!/(j+N+1)!$, $B(x,y)$ denoting the beta function. Hence

$$(2.21)\quad \int_\alpha^{1-k\alpha} y^{m-1}(1-k\alpha-y)^N dy = \sum_{j=0}^{m-1}\binom{m-1}{j}\alpha^{m-1-j}\frac{j!N!}{(j+N+1)!} \cdot (1-(k+1)\alpha)^{N+j+1}.$$

For $1 \leqq j \leqq m-1$, $(n-1)\binom{n-2}{m-1}\alpha^{m-1-j}N!/(j+N+1) = O((n\alpha)^{m-1-j}) = O((n\alpha)^{m-2})$. For $j = 0$, $(n-1)\binom{n-2}{m-1}\alpha^{m-1}N!/(N+1)! = (n\alpha)^{m-1}/(m-1)! + O((n\alpha)^{m-2})$. We conclude from these estimates and (2.21) that

$$(2.22)\quad (n-1)\binom{n-2}{m-1}\int_{\alpha}^{1-k\alpha} y^{m-1}(1-k\alpha)^N dy$$

$$= \left(\frac{(n\alpha)^{m-1}}{(m-1)!} + O((n\alpha)^{m-2}\right)\cdot(1-(k+1)\alpha)^{N+1}.$$

Substituting (2.22) into (2.19), we conclude

$$(2.23)\quad P(S_0^{(n)}, S_m^{(n)}, \cdots, S_{km}^{(n)} > \alpha) = \left(\frac{(n\alpha)^{(k+1)(m-1)}}{((m-1)!)^{k+1}} + O((n\alpha)^{(k+1)(m-1)-1})\right)$$

$$\cdot(1-(k+1)\alpha)^{n-1-(k+1)(m-1)}. \quad ■$$

Theorem (2.2) implies the following:

COROLLARY. *Let* $n = [(1/\alpha)(\log(1/\alpha) + m\log\log(1/\alpha) + x)]$. *For fixed* k, m,

$$P(S_0^{(n)}, S_m^{(n)}, \cdots, S_{(k-1)m}^{(n)} > \alpha) \sim \frac{e^{-kx}}{((m-1)!)^k}\,\frac{\alpha^k}{\log^k(1/\alpha)} \quad as\ \alpha \to 0.$$

PROOF. For α sufficiently small, the conditions $km \leqq n$, $(1/n) \leqq \alpha \leqq (1/k)$ are satisfied and Theorem 2.2 applies. Since $\log(1-k\alpha) = -k\alpha + O(\alpha^2)$ and $n = (1/\alpha)(\log(1/\alpha) + m\log\log(1/\alpha) + x) + O(1)$, we have

$$(2.24)\quad (n-1-k(m-1))\log(1-k\alpha) = k(\log\frac{1}{\alpha} + m\log\log\frac{1}{\alpha} + x) + O(\alpha\log\frac{1}{\alpha}).$$

Hence

$$(2.25)\qquad (1-k\alpha)^{n-1-k(m-1)} \sim e^{-kx}\left(\frac{1}{\alpha}\left(\log\frac{1}{\alpha}\right)^m\right)^{-k}.$$

Using $n \sim (1/\alpha)\log(1/\alpha)$, (2.13) and (2.25), we obtain

$$P(S_0^{(n)}, S_m^{(n)}, \cdots, S_{(k-1)m}^{(n)} > \alpha) \sim \frac{e^{-kx}}{((m-1)!)^k}\quad\frac{\alpha^k}{\log^k(1/\alpha)}.$$

REMARK. Suppose that $n(\alpha)$ is an integer valued function defined for $0 < \alpha < 1$ and satisfying $n(\alpha) = (1/\alpha)(\log(1/\alpha) + m\log\log(1/\alpha) + O(1))$. We conclude by the above method of proof that $P(S_0^{(n)}, S_m^{(n)}, \cdots, S_{(k-1)m}^{(n)} > \alpha) = O(\alpha^k/\log^k(1/\alpha))$.

We now proceed to estimate $P(S_{i_1}, \cdots, S \ > \alpha)$ in case $(i_1, \cdots, i_k)$ is not m-separated. We consider first a special case.

THEOREM 2.3. *Let* $1 \leqq l \leqq m-1$. *Let* $n = n(\alpha)$ *be a positive integer valued function of* α, $0 < \alpha < 1$, *satisfying* $n(\alpha) = (1/\alpha)(\log(1/\alpha) + m\log\log(1/\alpha) + O(1))$. *For fixed* l, m, *we have* $P(S_0, S_l > \alpha) = O(\alpha/\log^2\alpha)$.

PROOF. We may restrict ourselves to $0 < \alpha < \frac{1}{4}$. We have

(2.26) $[S_0, S_l > \alpha] \subseteq [Y_{l+m} \geqq 2\alpha] \cup [Y_m < \alpha,\ Y_{l+m} - Y_l > \alpha,\ Y_{l+m} < 2\alpha]$

where Y_j is the same as the quantity $Y_j^{(n)}$ introduced earlier. We estimate the probability of each event on the right side of (2.26). Using (2.14) with $S_0 = Y_m$ being replaced by Y_{l+m}, we obtain

$$P(Y_{l+m} > 2\alpha) = O((2n\alpha)^{l+m-1}(1-2\alpha)^n) = O((n\alpha)^{l+m-1} e^{-2n\alpha}). \tag{2.27}$$

Since $n\alpha = \log(1/\alpha) + m \log\log(1/\alpha) + O(1)$, we have

$$n\alpha = O\left(\log\frac{1}{\alpha}\right),\ e^{-n\alpha} = O\left(\frac{\alpha}{(\log(1/\alpha))^m}\right). \tag{2.28}$$

Substituting the estimates (2.28) in (2.27) and recalling that $l \leqq m-1$, we get

$$P(Y_{l+m} \geqq 2\alpha) = O\left(\frac{\alpha^2}{\log^2 \alpha}\right). \tag{2.29}$$

In view of (2.26) and (2.29), Theorem 2.3 will follow from the estimate $P(E) = O(\alpha/\log^2\alpha)$ where $E = [Y_m > \alpha, Y_{l+m} - Y_l > \alpha,\ Y_{l+m} < 2\alpha]$. We rewrite the latter as $E = [Y_l < Y_{l+m} - \alpha, \alpha < Y_m \leqq Y_{l+m}, \alpha < Y_{l+m} < 2\alpha]$. Let $f(x,y,z)$ denote the joint probability density function of the random variables Y_l, Y_m, Y_{l+m} It is known that

$$f(x,y,z) = \frac{(n-1)!}{[(l-1)!]^2(m-l-1)!(n-m-l-1)!} \cdot x^{l-1}(y-x)^{m-l-1}(z-y)^{l-1}(1-z)^{n-m-l-1}. \tag{2.30}$$

(For a derivation of (2.30) see [4, p. 387].) Hence

$$P(E) \leqq n^{m+l}\int_{\alpha}^{2\alpha} \int_{\alpha}^{z}\int_{0}^{z-\alpha} x^{l-1}(y-x)^{m-l-1}(z-y)^{l-1}(1-z)^{n-m-l-1}\, dxdydz. \tag{2.31}$$

Using the estimate $0 \leqq y-x \leqq 2\alpha$ and performing the integration in the x,y variables, (2.31) yields

$$P(E) \leqq \frac{2^{m-l-1}}{l^2} n^{m+l} \alpha^{m-l-1} \int_{\alpha}^{2\alpha} (z-\alpha)^{2l}(1-z)^{n-m-l-1}\, dz. \tag{2.32}$$

Now

$$\int_{\alpha}^{2\alpha}(z-\alpha)^{2l}(1-z)^{n-m-l-1}dz \leqq \int_{\alpha}^{1}(z-\alpha)^{2l}(1-z)^{n-m-l-1}\, dz$$
$$= \int_0^{1-\alpha} x^{2l}(1-\alpha-x)^{n-m-l-1}dx = (1-\alpha)^{n-m+l}\int_0^1 t^{2l}(1-t)^{n-m-l-1}dt.$$

The latter integral is recognized to be $B(2l+1, n-m-l) = (2l)!(n-m-l-1)!/(n-m+l)! = O(1/n^{2l+1})$. It follows that

$$(2.33)\qquad \int_{\alpha}^{2\alpha}(z-\alpha)^{2l}(1-z)^{n-m-l-1}dz = O\left(\frac{(1-\alpha)^n}{n^{2l+1}}\right) = O\left(\frac{e^{-n\alpha}}{n^{2l+1}}\right).$$

We conclude from (2.28), (2.32) and (2.33) that

$$(2.34)\qquad P(E) = O\left(\frac{\alpha}{\log^{l+1}(1/\alpha)}\right).$$

Since $l \geqq 1$, we have $P(E) = (\alpha/\log^2\alpha)$, thus proving Theorem 2.3. ∎

We now obtain an estimate for $P(S_{i_1},\cdots,S_{i_k} > \alpha)$ for all k-tuples $(i_1,\cdots,i_k)$ $(k \geqq 2)$ which fail to be m-separated. We introduce the notion of a component of $(i_1,\cdots,i_k)$. Let $A_{i_1},\ldots,_{i_k}$ be the union of the k open arcs $(p_{i_1},p_{i_1+m}),\cdots,(p_{i_k},p_{i_k+m})$ lying on C ($p_0,p_1,\ldots p_{n-1}$ are the points introduced in the proof of Theorem 2.1). Suppose that $A_{i_1},\cdots,_{i_k}$ decomposes into r components. Each component K consists of a number of the arcs (p_i,p_{i+m}). We shall say that the set I of indices i for which $(p_i,p_{i+m}) \in K$ forms a component of $(i_1,\cdots,i_k)$. Thus $(i_1,\cdots,i_k)$ will also decompose into r components. Clearly the number of components $r \leqq k$ and it is readily seen that $r = k$ if and only if $(i_1,\cdots,i_k)$ is m-separated.

We obtain an estimate for $P(S_{i_1},\cdots,S_{i_k} > \alpha)$ in terms of r.

THEOREM 2.4. *Let $C(r,k)$ be the class of all k-tuples $(i_1.\cdots,i_k)$ consisting of r components where $1 \leqq r < k$. Let $n = n(\alpha)$ be a positive integer-valued function defined on $(0,1)$ satisfying $n(\alpha) = (1/\alpha)(\log(1/\alpha) + \log\log(1/\alpha) + O(1))$. Then*

$$\max_{(i_1,\ldots,i_k)\in C(r,k)} P(S_{i_1},\cdots,S_{i_k} > \alpha) = O\left(\frac{\alpha^r}{\log^{r+1}(1/\alpha)}\right).$$

PROOF. The proof proceeds by induction on r. Suppose $r = 1$. The union of the k arcs $(p_{i_1},p_{i_1+m}),\cdots,(p_{i_k},p_{i_k+m})$ is connected. For $n \geqq m$, this implies that at least two of these arcs overlap, say (p_{i_j},p_{i_j+m}) and (p_{i_k},p_{i_k+m}). This means that S_{i_j} and S_{i_k} have common summands. Since the L_i's are exchangeable, we have $P(S_{i_j},S_{i_k} > \alpha) = P(S_0,S_l > \alpha)$ for some $1 \leqq l \leqq m-1$. Hence $P(S_{i_1},\cdots,S_{i_k} > \alpha) \leqq \max_{1\leqq l\leqq m-1} P(S_0,S_l)$ for all $(i_1,\cdots,i_k)$ for which $r = 1$. Hence for $r = 1$, Theorem 2.4 is a direct consequence of Theorem 2.3.

Suppose that Theorem 2.4 holds for $r-1$. We show that it holds for r. Suppose first that some component of $(i_1,\cdots,i_k)$ consists of just one index, say i_j. Since the L_i's are exchangeable, we may assume that $j = 1$ and $i_1 = 0$. For we have $P(S_{i_1},\cdots,S_{i_k} > \alpha) = P(S_0,S_{i_{j+1}-i_j},\cdots,S_{i_k-i_j},S_{i_1+n-i_j},\cdots,S_{i_{j-1}+n-i_j} > \alpha$ and $\{0\}$ is a component of the k-tuple $(0,i_{j+1}-i_j,\cdots,i_k-i_j,i_1+n-i_j,\cdots,i_{j-1}$

$+ n - i_j)$. Since $\{0\}$ is a component of $(0, i_2, \cdots, i_k)$, we have $m \leqq i_2$ and $i_k \leqq n - m$. We may therefore employ the lemma preceding Theorem 2.2. We obtain

$$P(S_0^{(n)}, S_{i_2}^{(n)}, \cdots, S_{i_k}^{(n)} > \alpha) = \int_\alpha^1 P\left(S_{i_2-m}^{(n-m)}, \cdots, S_{i_k-m}^{(n-m)} > \frac{\alpha}{1-y}\right) dF(y)$$

(2.35)

$$\leqq P(S_{i_2-m}^{(n-m)}, \cdots, S_{i_k-m}^{(n-m)} > \alpha) \cdot P(S_0^{(n)} > \alpha).$$

The $(k-1)$ tuple $(i_2 - m, \cdots, i_k - m)$ consists of $(r-1)$ components. Since $r - 1 < k - 1$, we conclude from the induction hypothesis that $P(S_{i_2-m}^{(n-m)}, \cdots, S_{i_k-m}^{(n-m)} > \alpha) \leqq C\alpha^{r-1}/\log^r(1/\alpha)$ for some positive C and for all choices of $i_2, \cdots, i_k$. Furthermore we have $P(S_0^{(n)} > \alpha) = O(\alpha/\log(1/\alpha))$ (see the remark following the Corollary of Theorem 2.2). Hence

$$\max_{(i_1, \cdots, i_k) \in C_1(r,k)} P(S_{i_1}, \cdots, S_{i_k} > \alpha) = O\left(\frac{\alpha^r}{\log^{r+1}(1/\alpha)}\right),$$

$C_1(r,k)$ consists of those k-tuples in $C(r,k)$ containing a component consisting of one index. This easily yields the general result, for let $(i_1, \cdots, i_k)$ consist of r components, $r \geqq 2$, and suppose that each component has more than one index. Remove all but one index from one of the components and let $i_1' < \cdots < i_s'$ denote the remaining indices. Then r = number of components of $(i_1', \cdots, i_s')$. As each component of $(i_1, \cdots, i_s)$ has more than one index, we have $r < s$. Since

$$P(S_{i_1}, \cdots, S_{i_k} > \alpha) \leqq P(S_{i_1}, \cdots, S_{i_s} \geqq \alpha) \leqq \max_{(i_1 \cdots i_s) \in C_1(r,s)} = O\left(\frac{\alpha^r}{\log^{r+1}(1/\alpha)}\right)$$

we obtain

$$\max_{(i_1, \cdots, i_k) \in C(r,k)} P(S_{i_1}, \cdots, S_{i_k} > \alpha) = O\left(\frac{\alpha^r}{\log^{r+1} 1/\alpha)}\right) . \blacksquare$$

The Corollary to Theorem 2.2 and Theorem 2.4 yield

THEOREM 2.5. *Let* $\cap = [(1/\alpha)(\log(1/\alpha) + m \log\log(1/\alpha) + x)]$. *Then*

$$\lim_{\alpha \to 0} \sum_{0 \leqq i_1 < i_2 < \cdots < i_k \leqq n-1} P(S_{i_1}, \cdots, S_{i_k} > \alpha) = \frac{1}{k!} \frac{e^{-kx}}{((m-1)!)^k}.$$

PROOF. We write the above sum Σ as $\Sigma = \Sigma_1 + \Sigma_2$ where Σ_1 extends over the k-tuples $(i_1, \cdots, i_k)$ which are m-separated, and Σ_2 extends over all other k-tuples.

Let $n_{k,r}$ be the number of k-tuples $(i_1, \ldots, i_k)$ consisting of r components. Thus

$n_{k,k}$ = number of k-tuples which are m-separated. Let $I_1,\cdots,I_r$ denote the r components of $(i_1,\ldots,i_k)$. Let i'_j be any index in I_j, $1 \leqq j \leqq r$. It is clear that for a given choice of $i'_1,\cdots,i'_r$, the number of choices of the remaining indices $\leqq C$, where C is a positive constant depending only on k,m. Since the number of possible choices of $i'_1,\cdots,i'_r = O(n^r)$, we obtain $n_{k,r}= O(n^r)$. The total number of k-tuples $(i_1,\cdots,i_k) = \binom{n}{k} = n^k/k! + O(n^{k-1})$. Hence

$$n_{kk} = \binom{n}{k} - \sum_{r=1}^{k-1} n_{kr} = \binom{n}{k} + O(n^{k-1}) \sim \frac{1}{k!}\left(\frac{1}{\alpha}\log\frac{1}{\alpha}\right)^k . \tag{2.36}$$

Now $\Sigma_1 = n_{kk}P(S_0,S_m,\cdots,S_{(k-1)m} > \alpha)$. It follows from the Corollary to Theorem 2.2 and (2.36) that

$$\lim_{\alpha\to 0} \Sigma_1 = \frac{e^{-kx}}{k!((m-1)!)^k}. \tag{2.37}$$

which implies Theorem 2.5, provided we can show $\lim_{\alpha\to 0}\Sigma_2 = 0$. We write $\Sigma_2 = \Sigma_{2,1} + \cdots + \Sigma_{2,k-1}$ where $\Sigma_{2,r}$ is the sum $\Sigma P(S_{i_1},\cdots,S_{i_k} > \alpha$ extending over the k-tuples $(i_1,\cdots,i_k)$ with r components. Using Theorem 2.4 and bearing in mind that $n_{k,r} = O(n^r) = O((1/\alpha\log 1/\alpha)^k)$, we have

$$\Sigma_{2,r} = O\left(\frac{1}{\log(1/\alpha)}\right) \quad , \quad 1 \leqq r \leqq k-1. \tag{2.38}$$

Hence $\lim_{\alpha\to 0}\Sigma_{2,r} = 0$ for $1 \leqq r \leqq k-1$. It follows that $\lim_{\alpha\to 0}\Sigma_2 = 0$, thus proving Theorem 2.5. ■

We now state the main result of this paper.

THEOREM 2.6. *Let $N_{\alpha,m}$ equal the number of randomly placed arcs of length α required to cover the circle C m times. Then*

$$\lim_{\alpha\to 0} P\left(N_{\alpha,m} \leqq \frac{1}{\alpha}\left(\log\frac{1}{\alpha} + m\log\log\frac{1}{\alpha} + x\right)\right) = e^{-1/m-1))!e^{-x}} .$$

PROOF. We have $P(N_{\alpha,m} \leqq (1/\alpha)(\log(1/\alpha) + m\log\log(1/\alpha) + x)) = P(N_{\alpha,m} \leqq n)$ where $n = [(1/\alpha)(\log(1/\alpha) + m\log\log(1/\alpha) + x)]$. Using the inclusion-exclusion inequalities (2.1) and Theorem 2.5, we get

$$\lim_{\alpha\to 0} P\left(N_{\alpha,m} \leqq \frac{1}{\alpha}\right)\log\frac{1}{\alpha} + m\log\log\frac{1}{\alpha} + x\Big)\Big) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\left(\frac{e^{-x}}{(m-1)!}\right)^k = e^{-1/(m-1)!e^{-x}}.$$

3.

We show in the present section that Theorem 2.6 yields Steutel's asymptotic formula (1.4) for $E(N_{\alpha,1})$. We prove

THEOREM 3.1. *Let* $F_\alpha(x) = P(N_{\alpha,1} \leqq (1/\alpha)(\log(1/\alpha) + \log\log(1/\alpha) + x))$.

i) *For* $0 < \alpha < e^{-1}$ *and* $x \geqq 1$ *we have* $1 - F_\alpha(x) \leqq Cxe^{-x}$, *C being a positive constant independent of* α.

ii) *A similar statement holds for* $x \leqq -1$, $F_\alpha(x) \leqq C|x|e^{-|x|}$ *for* $0 < \alpha < e^{-2}$ *and* $x \leqq -1$, *C being a positive independent of* α.

Before proving Theorem 3.1, we show how it yields Steutel's formula for $E(N_{\alpha,1})$. We have $E(N_{\alpha,1}) = (1/\alpha)(\log(1/\alpha) + \log\log(1/\alpha) + \int_{-\infty}^{\infty} x dF_\alpha(x))$, so that we must show $\lim_{\alpha\to 0}\int_{-\infty}^{\infty} x dF_\alpha(x) = \gamma$. For any $R \geqq 1$, we write

$$\int_{-\infty}^{\infty} x dF_\alpha(x) = \int_{-\infty}^{-R} x dF_\alpha(x) + \int_{-R}^{R} x dF_\alpha(x) + \int_{R}^{\infty} x dF_\alpha(x). \tag{3.1}$$

Since $\lim_{\alpha\to 0} F_\alpha(x) = e^{-e^{-x}}$ we have $\lim_{\alpha\to 0}\int_{-R}^{R} x dF_\alpha(x) = \int_{-R}^{R} x d(e^{-e^{-x}})$. Now $\int_R^\infty x dF_\alpha(x) = -\int_R^\infty x d(1 - F_\alpha(x)) = R(1 - F_\alpha(R)) + \int_R^\infty (1 - F_\alpha(x))dx$. Using Theorem 3.1(i), we conclude that $\int_R^\infty x dF_\alpha(x) \leqq CR^2e^{-R} + C\int_R^\infty xe^{-x}dx$. It follows that $\int_R^\infty x dF_\alpha(x) \to 0$ as $R \to \infty$ uniformly for $0 < \alpha < e^{-1}$. Similarly Theorem 3.1 (ii) implies that $\int_{-\infty}^{-R} x dF_\alpha(x) \to 0$ as $R \to \infty$ uniformly for $0 < \alpha < e^{-2}$. We conclude readily from (3.1) that $\lim \int_{-\infty}^{\infty} x dF_\alpha(x) = \int_{-\infty}^{\infty} x d(e^{-e^{-x}})$. Letting $t = e^{-x}$, we get $\int_{-\infty}^{\infty} x d(e^{-e^{-x}}) = \int_{-\infty}^{\infty} xe^{-e^{-x}}e^{-x}dx = -\int_0^\infty \log t\, e^{-t}dt$. The latter integral is recognized to be $-\Gamma'(1)$ and it is known that $-\Gamma'(1) = \gamma$[1]. Hence $\lim_{\alpha\to 0}^{\infty}\int_{-\infty}^{\infty} x dF_\alpha(x) = \gamma$, proving Steutel's formula (1.4).

PROOF OF THEOREM 3.1.

i) $1 - F_\alpha(x) = P(N_{\alpha,1} > n)$ where $n = [(1/\alpha)(\log(1/\alpha) + \log\log(1/\alpha) + x)]$. Now $P(N_{\alpha,1} > n) = P(\bigcup_{i=0}^{n-1}(L > \alpha)) \leqq \sum_{i=0}^{n-1} P(L_i > \alpha)$. $P(L_i) = P(L_0) = (1-\alpha)^{n-1}$ (formula (2.2)). Hence

$$P(N_{\alpha,1} > n) \leqq n(1-\alpha)^{n-1} \leqq 2ne^{-n\alpha}. \tag{3.2}$$

Since

$$\log\frac{1}{\alpha} + \log\log\frac{1}{\alpha} + x - \alpha \leqq n\alpha \leqq \log\frac{1}{\alpha} + \log\log\frac{1}{\alpha} + x, \tag{3.3}$$

we obtain

$$P(N_{\alpha,1} > n) \leqq 2\frac{\left(\log\frac{1}{\alpha} + \log\log\frac{1}{\alpha} + x\right)}{\log 1/\alpha}e^{\alpha - x} \leqq 2\left(2 + \frac{x}{\log 1/\alpha}\right)e^{1-x} \leqq 6exe^{-x} \tag{3.4}$$

the last inequality of which is valid for $0 < \alpha < e^{-1}$, $x \geqq 1$. Thus Theorem 3.1(i) is established with the choice $C = 6e$.

ii) The proof of Theorem 3.1(ii) is based on the following lower bound for $P(N_{\alpha,1} > n)$ obtained in [7, Formula 91]. We assume in the sequel that $0<\alpha<e^{-2}$ $x \leqq -1$.

$$(3.5)\qquad P(N_{\alpha,1} > n) \geqq \frac{(1-\alpha)^{2n}}{(1-2\alpha)^{n+1} + \frac{2}{n+1}[(1-\alpha)^{n+1} - (1-2\alpha)^{n+1}]}.$$

The denominator on the right side of (3.5) $\leqq (1-\alpha)^{2n} + 2/n(1-\alpha)^n$. We conclude from (3.5) that $P(N_{\alpha,1} > n) \geqq n(1-\alpha)^n/(n(1-\alpha)^n + 2)$. Hence

$$(3.6)\qquad P(N_{\alpha,1} \leqq n) \leqq \frac{2}{n(1-\alpha)^n}.$$

Let $n = [(1/\alpha)(\log(1/\alpha) + \log\log(1/\alpha) + x)]$ where $x < 0$. We observe that $F_\alpha(x) = P(N_{\alpha,1} \leqq n) = 0$ for $n\alpha < 1$, in which case Theorem 3.1(ii) is obviously true. We therefore assume $n\alpha \geqq 1$ so that $x \geqq -\log(1/\alpha) - \log\log(1/\alpha) + 1$.

Now

$$(3.7)\qquad n(1-\alpha)^n \geqq ne^{-n\alpha - n\alpha^2}.$$

Using (3.3) we easily verify the inequalities

$$(3.8)\quad n \geqq \frac{1}{2\alpha}\left(\log\frac{1}{\alpha} + \log\log\frac{1}{\alpha} + x\right) e^{-n\alpha} \geqq \frac{\alpha e^{-x}}{\log 1/\alpha}, e^{-n\alpha^2} \geqq e^{-2(1/\alpha)\log(1/\alpha)} \geqq e^{-4e^2}$$

which are valid for $-\log(1/\alpha) - \log\log(1/\alpha) + 1 \leqq x \leqq 0$, $0 < \alpha < e^{-2}$. We obtain from (3.7) and (3.8)

$$(3.9)\quad n(1-\alpha)^n \geqq \frac{e^{4e^{-2}}}{2} f(x)\, \frac{e^{|x|}}{|x|},\quad -\log\frac{1}{\alpha} - \log\log\frac{1}{\alpha} + 1 \leqq x \leqq 0 \text{ and } 0<\alpha<e^{-2},$$

where $f(x) = -x(\log(1/\alpha) + \log\log(1/\alpha) + x)/\log 1/\alpha$.

The minimum of $f(x)$ on $[-\log(1/\alpha) + \log\log(1/\alpha) + 1, -1]$ is attained at -1. Hence $f(x) \geqq f(-1) = (\log(1/\alpha) + \log\log(1/\alpha) - 1)/\log(1/\alpha) \geqq 1 - 1/\log(1/\alpha)$, the latter being $\geqq \frac{1}{2}$ for $0 < \alpha < e^{-2}$. Hence

$$(3.10)\qquad n(1-\alpha)^n \geqq \frac{e^{-4e^{-2}}}{4}\,\frac{e^{|x|}}{|x|},\quad x \leqq -1 \text{ and } 0 < \alpha < e^{-2}.$$

We conclude from (3.6) and (3.10) that

$$(3.11)\qquad F_\alpha(x) \leqq C|x|e^{-|x|},\quad x \leqq -1 \text{ and } 0 < \alpha < e^{-2}$$

with the choice $C = 8e^{4e^{-2}}$, thus proving Theorem 3.1(ii). ∎

We conclude by mentioning two open problems suggested by the present paper.

1) It seems likely that Steutel's formula (1.4) should generalize as follows for all m:

$$E(N_{\alpha,m}) = \frac{1}{\alpha}\left(\log\frac{1}{\alpha} + \log\log\frac{1}{\alpha} + \gamma_m + o(1)\right) \text{ as } \alpha \to 0 \tag{3.12}$$

where $\gamma_m = \gamma - \log\log(m-1)!$

As in the case $m = 1$, (3.12) follows from the formal step

$$\lim_{\alpha\to 0}\int_{-\infty}^{\infty} x dF_{\alpha,m}(x) = \int_{-\infty}^{\infty} x d(e^{-1/(m-1)!\, e^{-x}}$$

where $F_{\alpha,m}(x) = P(N_{\alpha.m} \leqq (1/\alpha)(\log(1/\alpha) + m\log\log(1/\alpha) + x))$. The justification of the formal step would follow from a generalization of Theorem 3.1 to $m > 1$, which would in turn follow from a suitable generalization of 3.5 to $m > 1$.

2) The problem of random coverings of C has the following discrete analog considered by Erdös and Renyi [2]. Balls are thrown into n urns uniformly and independently. Let $N'_{n,m}$ equal the number of throws necessary to obtain at least m balls in each urn. It is shown in [1] that

$$\lim_{n\to\infty} P(N'_{n,m} \leqq n(\log n + (m-1)\log\log n + x)) = e^{-1/(m-1)!\, e^{-x}} \quad .$$

This result is a discrete analog of Theorem 2.6 with α being replaced by $1/n$.

The Erdös-Renyi result may be reformulated in the context of our problem. Let $N'_{n,m}$ equal the number of randomly placed arcs of length $1/n$ necessary to cover m times a given lattice L of n equally spaced points on the circle C. It is readily seen that the random variable $N'_{n,m}$ is essentially the same as the $N'_{n,m}$ considered by Erdös and Renyi. We merely replace the words ball and urn by arc of length $1/n$ and point of L. Hence

$$\lim_{n\to\infty} P(N'_{n,m} \leqq n(\log n + (m-1)\log\log n + x)) = e^{-1/(m-1)!\, e^{-x}} \quad .$$

On the other hand, setting $N_{n,m} = N_{1/n,m}$, Theorem 2.6 states that

$$\lim_{n\to\infty} P(N_{n,m} \leqq n(\log n + m\log\log n + x)) = e^{-1/(m-1)!\, e^{-x}} \ .$$

Comparing the two results, one readily verifies that $N''_{n,m}/n\log\log n \to 1$ in measure as $n \to \infty$. Here $N''_{n,m} = N_{n,m} - N'_{n,m}$ = number of arcs of length $1/n$ which must be thrown to cover C m times after L has already been covered m times. It is reasonable to conjecture that $(N''_{n,m}/n - \log\log n)$ has a proper limiting distribution as $n \to \infty$, but we have not been able to prove this.

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